2012 MATH 7402

1. (a) To find a solution of

$$xy'' + (1-x)y' + \lambda y = 0, \qquad \lambda \text{ constant.}$$

insert the Frobenius ansatz

$$y(x) = \sum_{k=0}^{\infty} a_k x^{k+c}, \qquad (a_0 = 1)$$

to give

$$\sum_{k=0}^{\infty}(k+c)(k+c-1)a_k x^{k+c-1} + \sum_{k=0}^{\infty}(k+c)a_k x^{k+c-1} - \sum_{k=0}^{\infty}(k+c)a_k x^{k+c} + \sum_{k=0}^{\infty}\lambda a_k x^{k+c} = 0.$$

Summing the third and fourth terms and reindexing downwards gives the single power series

$$\sum_{k=0}^{\infty}\left(a_k(k+c)^2 - a_{k-1}(k+c-1-\lambda)\right)x^{k+c-1} = 0,$$

where the convention $a_{-1} = 0$ holds. Setting the coefficients to zero gives the indicial equation $c^2 = 0$ for $k = 0$ and the recurrence relation

$$a_k = a_{k-1}\frac{(k-1+c-\lambda)}{(k+c)^2},$$

for $k \geq 1$. The solution for a_k can be written as an expression involving products of k terms

$$a_k = \frac{(k-1+c-\lambda)(k-2+c-\lambda)...(c-\lambda)}{(k+c)^2(k-1+c)^2...(1+c)^2} = \frac{\Gamma(k+c-\lambda)\,\Gamma(c+1)^2}{\Gamma(c-\lambda)\,\Gamma(k+c+1)^2}$$

where the Gamma function identity

$$(k+c)(k-1+c)...(1+c) = \frac{\Gamma(k+c+1)}{\Gamma(c+1)}$$

has been used to express the products in terms of the Gamma function. Inserting $c = 0$ into the expression for a_k, and using $\Gamma(k+1) = k!$ for integer $k \geq 0$, gives the first solution

$$y_1(x) = \sum_{k=0}^{\infty} a_k(0)x^k = \sum_{k=0}^{\infty}\frac{\Gamma(k-\lambda)}{\Gamma(-\lambda)(k!)^2}x^k.$$

as required.

(b) The above formula fails when $\lambda = n \geq 0$ integer because $\Gamma(-n)$ in the denominator is not defined. Instead, what happens is that for $k = n+1$ the recurrence relation has the form

$$a_{n+1} = a_n\frac{(n-\lambda)}{(n+1)^2} = 0, \qquad \text{as } \lambda = n.$$

The solution is therefore a polynomial of order n as only $\{a_0, a_1, ..., a_n\}$ are non-zero. For $k \leq n$ the recurrence relation is

$$a_k = a_{k-1}\frac{(k-1-n)}{k^2},$$

from which we can deduce that

$$a_k = \frac{(-n)(-n+1)...(-n+k-1)}{(k!)^2} = \frac{(-1)^k}{(k!)}\begin{pmatrix} n \\ k \end{pmatrix},$$

since

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{(n)(n-1)...(n-k+1)}{(k!)}.$$

Hence we have the Laguerre polynomial result

$$L_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k!} x^k.$$

(c) Leibniz rule provides the following formula for differentiating a product n times

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}.$$

Writing and $f(x) = x^n$ and $g(x) = e^{-x}$ gives (think carefully about differentiating x^n exactly $(n-k)$ times!)

$$f^{(n-k)}(x) = n(n-1)...(k+1)x^k = \frac{n!}{k!}x^k, \quad \text{and} \quad g^{(k)}(x) = (-1)^k e^{-x}.$$

Inserting these into the Leibniz rule formula gives

$$L_n(x) = \frac{e^x}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^n e^{-x}\right) = \frac{e^x}{n!}\sum_{k=0}^{n} \binom{n}{k} \frac{n!}{k!}x^k(-1)^k e^{-x} = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k!} x^k.$$

as above.

2. To seek a solution separate variables $u(r,\theta) = R(r)T(\theta)$ and insert in Laplace's equation to give

$$\frac{T}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{R}{r^2}\frac{d^2T}{d\theta^2} = 0, \quad \text{or} \quad \frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = \frac{1}{T}\frac{d^2T}{d\theta^2} = -\lambda \ \text{(sep. cons.)}$$

Looking at the T-equation first, the boundary conditions at $\theta = 0, \beta$ imply that $T(0) = T(\beta) = 0$ and

$$\frac{d^2T}{d\theta^2} + \lambda T = 0, \quad T(0) = T(\beta) = 0$$

is a regular Sturm-Liouville eigenvalue problem on $(0, \beta)$. Examining the general solution

$$T(\theta) = \begin{cases} A\cos\sqrt{\lambda}\theta + B\sin\sqrt{\lambda}\theta & \lambda > 0 \\ C\theta + D & \lambda = 0 \\ E\cosh\sqrt{-\lambda}\theta + F\sinh\sqrt{-\lambda}\theta & \lambda < 0 \end{cases}$$

it is evident that eigenvalues are only possible for $\lambda > 0$. The b.c. $T(0) = 0$ gives $A = 0$ and $T(\beta) = 0$ requires $\sin\sqrt{\lambda}\beta = 0$ for a non-zero solution. Hence we have eigenvalues and eigenfunctions

$$\lambda_k = \frac{k^2\pi^2}{\beta^2} \quad k \geq 1 \text{ integer}, \quad \text{and} \quad T_k(\theta) = \sin\left(\frac{k\pi\theta}{\beta}\right).$$

Next, forming the R-equation for $\lambda = \lambda_k$ gives

$$r\frac{d}{dr}\left(r\frac{dR_k}{dr}\right) - \frac{k^2\pi^2}{\beta^2}R_k = 0$$

Trying a solution $R_k = r^p$ results in

$$\left(p^2 - \frac{k^2\pi^2}{\beta^2}\right)r^p = 0, \quad \Longrightarrow \quad R_k(r) = C_k r^{k\pi/\beta} + D_k r^{-k\pi/\beta}.$$

Temperature must be finite at $r = 0$, so all $D_k = 0$, and we have the general solution

$$u(r,\theta) = \sum_{k=1}^{\infty} R_k(r)T_k(\theta) = \sum_{k=1}^{\infty} C_k r^{k\pi/\beta} \sin\left(\frac{k\pi\theta}{\beta}\right)$$

as required. To find the $\{C_k\}$ we need to use the final b.c. $u(a,\theta) = h(\theta)$, or

$$\sum_{k=1}^{\infty} C_k a^{k\pi/\beta} T_k(\theta) = h(\theta).$$

Use the orthogonality condition (c.f. sine Fourier series),

$$\langle T_k, T_j\rangle = \int_0^{\beta} \sin\left(\frac{k\pi\theta}{\beta}\right) \sin\left(\frac{j\pi\theta}{\beta}\right) \, \mathrm{d}\theta = \begin{cases} 0 & j \neq k \\ \beta/2 & j = k \end{cases}$$

to take the inner product of both sides with T_j, resulting in

$$(\beta/2) C_j a^{j\pi/\beta} = \langle h, T_j\rangle = \int_0^{\beta} h(\theta) \sin\left(\frac{j\pi\theta}{\beta}\right) \, \mathrm{d}\theta$$

or

$$C_j = \frac{2}{\beta a^{j\pi/\beta}} \int_0^{\beta} h(\theta) \sin\left(\frac{j\pi\theta}{\beta}\right) \, \mathrm{d}\theta, \quad \text{giving} \quad \alpha_j = \frac{2}{\beta a^{j\pi/\beta}}.$$

3. (a) Evaluating Laplace transforms

(i) $\mathcal{L}[e^{-at}] = \int_0^{\infty} e^{-(s+a)t} \, \mathrm{d}t = \frac{1}{s+a}$ valid for $\mathrm{Re}(s+a) > 0$.

(ii) $\mathcal{L}[t^{\alpha}] = \int_0^{\infty} t^{\alpha} e^{-st} \, \mathrm{d}t = \int_0^{\infty} \left(\frac{q}{s}\right)^{\alpha} e^{-q} \, \frac{\mathrm{d}q}{s} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$ valid for $\alpha \neq$ -ve integer, (using $q = st$).

(iii) $\mathcal{L}[g(t)] = \int_0^{\infty} f(t)e^{-(s+b)t} \, \mathrm{d}t = \bar{f}(s+b)$ valid for $\mathrm{Re}(s+b) > 0$

(iv) $\mathcal{L}[h(t)] = \int_c^{\infty} f(t-c)e^{-st} \, \mathrm{d}t = \int_0^{\infty} f(q)e^{-s(q+c)} \, \mathrm{d}q = e^{-sc}\bar{f}(s)$ using $q = t - c$

(b) First of all note that

$$t^{\alpha} \cosh(\beta t) \sin(\gamma t) = \frac{1}{4\mathrm{i}} t^{\alpha} \left(e^{(\beta+\gamma \mathrm{i})t} - e^{(\beta-\gamma \mathrm{i})t} + e^{(-\beta+\gamma \mathrm{i})t} - e^{(-\beta-\gamma \mathrm{i})t}\right).$$

Now use result (iii) with $f(t) = t^{\alpha}$

$$\mathcal{L}[t^{\alpha} \cosh(\beta t) \sin(\gamma t)] = \frac{1}{4\mathrm{i}} \left(\bar{f}(s-\beta-\gamma \mathrm{i}) - \bar{f}(s-\beta+\gamma \mathrm{i}) + \bar{f}(s+\beta-\gamma \mathrm{i}) - \bar{f}(s+\beta+\gamma \mathrm{i})\right)$$

$$= \frac{\Gamma(\alpha+1)}{4\mathrm{i}} \left(\frac{1}{(s-\beta-\gamma \mathrm{i})^{\alpha+1}} - \frac{1}{(s-\beta+\gamma \mathrm{i})^{\alpha+1}} + \frac{1}{(s+\beta-\gamma \mathrm{i})^{\alpha+1}} - \frac{1}{(s+\beta+\gamma \mathrm{i})^{\alpha+1}}\right)$$

using the result (ii). From (iii) above, need Re $s > \beta$ and Re $s > -\beta$ for validity, i.e. Re $s > |\beta|$.

(c) The case $\alpha = 1$ simplifies, using $\Gamma(2) = 1! = 1$, to

$$\begin{aligned}
\bar{f}(s) &= \frac{1}{4\mathrm{i}}\left(\frac{1}{(s-\beta-\gamma\mathrm{i})^2} - \frac{1}{(s-\beta+\gamma\mathrm{i})^2} + \frac{1}{(s+\beta-\gamma\mathrm{i})^2} - \frac{1}{(s+\beta+\gamma\mathrm{i})^2}\right) \\
&= \frac{1}{4\mathrm{i}}\left(\frac{(s-\beta+\gamma\mathrm{i})^2-(s-\beta-\gamma\mathrm{i})^2}{(s-\beta-\gamma\mathrm{i})^2(s-\beta+\gamma\mathrm{i})^2} + \frac{(s+\beta+\gamma\mathrm{i})^2-(s+\beta-\gamma\mathrm{i})^2}{(s+\beta-\gamma\mathrm{i})^2(s+\beta+\gamma\mathrm{i})^2}\right) \\
&= \frac{1}{4\mathrm{i}}\left(\frac{4\mathrm{i}\gamma(s-\beta)}{((s-\beta)^2+\gamma^2)^2} + \frac{4\mathrm{i}\gamma(s+\beta)}{((s+\beta)^2+\gamma^2)^2}\right)
\end{aligned}$$

as required.

4. To solve

$$x^2(x+1)y'' + x(6x+3)y' + (6x+1)y = 0$$

insert the Frobenius ansatz

$$y(x,c) = \sum_{k=0}^{\infty} a_k(c)x^{k+c}, \qquad a_0 = 1,$$

to give

$$\sum_{k=0}^{\infty} a_k(k+c)(k+c-1)x^{k+c+1} + \sum_{k=0}^{\infty} a_k(k+c)(k+c-1)x^{k+c} + \sum_{k=0}^{\infty} 6a_k(k+c)x^{k+c+1}$$
$$+\sum_{k=0}^{\infty} 3a_k(k+c)x^{k+c} + \sum_{k=0}^{\infty} 4a_k x^{k+c+1} + \sum_{k=0}^{\infty} a_k x^{k+c} = 0.$$

(a) Re-indexing the first, third and fifth terms (down one) gives a single power series

$$\sum_{k=0}^{\infty}\Big(a_k\left((k+c)(k+c-1)+3(k+c)+1\right)$$
$$+\, a_{k-1}\left((k+c-1)(k+c-2)+6(k+c-1)+6\right)\Big)x^{k+c} = 0$$

where the usual convention $a_{-1} = 0$ is adopted. Setting the coefficients in the power series to zero, gives the indicial equation for $k = 0$

$$c(c-1)+3c+1 = 0, \quad \text{or} \quad c = -1, -1.$$

and the recurrence relation for $k \geq 1$

$$a_k = -\frac{(k+c+1)(k+c+2)}{(k+c+1)^2}a_{k-1} = -\frac{(k+c+2)}{(k+c+1)}a_{k-1},$$

as required.

(b) Expansion of the recurrence relation reveals that

$$a_k(c) = (-1)^k\frac{(k+c+2)(k+c+1)\ldots(c+3)}{(k+c+1)(k+c)\ldots(c+2)}a_0 = (-1)^k\frac{k+c+2}{c+2}.$$

(using the convention that $a_0 = 1$). Inserting $c = -1$ yields the solution

$$y_1(x) = \sum_{k=0}^{\infty} a_k(-1)x^{k-1} = \sum_{k=0}^{\infty}(-1)^k(k+1)x^{k-1} = \frac{1}{x} - 2 + 3x - 4x^2 \ldots.$$

(c) The second solution is obtained by first differentiating y w.r.t. c

$$\frac{\partial y}{\partial c} = \log x \sum_{k=0}^{\infty} a_k(c)x^{k+c} + \sum_{k=1}^{\infty} \frac{\mathrm{d}a_k}{\mathrm{d}c}(c)x^{k+c}, \quad \text{and noting that} \quad \frac{\mathrm{d}a_k}{\mathrm{d}c}(c) = \frac{(-1)^{k+1}k}{(c+2)^2}.$$

Evaluating at $c = -1$ gives the second solution

$$y_2(x) = y_1(x)\log x + \sum_{k=1}^{\infty}(-1)^{k+1}kx^{k-1} = y_1(x)\log x + \left(1 - 2x + 3x^2 - 4x^3 + \ldots\right),$$

as required.

5. (a) The Fourier transform of a convolution is

$$\begin{aligned}
\mathcal{F}[(f*g)] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(x-y)g(y)\ \mathrm{d}y\right)e^{-ikx}\ \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-y)g(y)e^{-ikx}\ \mathrm{d}x\,\mathrm{d}y \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)g(y)e^{-ik(v+y)}\ \mathrm{d}v\,\mathrm{d}y \qquad (\text{writing } v = x-y) \\
&= \sqrt{2\pi}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v)e^{-ikv}\ \mathrm{d}v\right)\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(y)e^{-iky}\ \mathrm{d}y\right) \\
&= \sqrt{2\pi}\hat{f}(k)\hat{g}(k)
\end{aligned}$$

(b) Taking the x-transform of the equation and boundary conditions, and applying the result

$$\mathcal{F}[f'(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(x)e^{-ikx}\ \mathrm{d}x = \frac{1}{\sqrt{2\pi}}\left(\left[f(x)e^{-ikx}\right]_{-\infty}^{\infty} + ik\int_{-\infty}^{\infty} f'(x)e^{-ikx}\ \mathrm{d}x\right) = (ik)\hat{f}(k),$$

twice to obtain $\mathcal{F}[u_{xx}] = (ik)^2\hat{u} = -k^2\hat{u}$, leads to

$$-k^2\hat{u} + \hat{u}_{yy} = 0, \quad \hat{u}(k,0) = \hat{f}(k), \quad \hat{u}(k,y) \to 0, \quad \text{as} \quad y \to \infty.$$

Integrating the equation gives

$$\hat{u}(k,y) = A(k)e^{ky} + B(k)e^{-ky}, \quad \text{or equivalently} \quad \hat{u}(k,y) = \tilde{A}(k)e^{|k|y} + \tilde{B}(k)e^{-|k|y}.$$

The boundary conditions can be applied to the second form to give $\tilde{A}(k) = 0$, $\tilde{B}(k) = \hat{f}(k)$, giving

$$\hat{u}(k,y) = \hat{f}(k)e^{-|k|y} = \sqrt{2\pi}\hat{f}(k)\hat{g}(k), \quad \text{for } \hat{g}(k) = \frac{e^{-|k|y}}{\sqrt{2\pi}}.$$

Clearly $u(x,y)$ can be obtained from the convolution theorem as

$$u(x,y) = \int_{-\infty}^{\infty} f(x-t)g(t)\ \mathrm{d}t,$$

and it remains to find $g(x)$ to complete the proof.

$$\begin{aligned}
g(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{g}(k)e^{ikx}\ \mathrm{d}k = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-|k|y}\ \mathrm{d}k = \frac{1}{\pi}\int_0^{\infty}\cos kx\, e^{-|k|y}\ \mathrm{d}k \\
&= \mathrm{Re}\left\{\frac{1}{\pi}\int_0^{\infty} e^{k(ix-y)}\ \mathrm{d}k\right\} = \mathrm{Re}\left\{\frac{1}{\pi(ix-y)}\left[e^{k(ix-y)}\right]_0^{\infty}\right\} = \mathrm{Re}\left\{\frac{1}{\pi(y-ix)}\right\} = \frac{y}{\pi}\frac{1}{x^2+y^2}
\end{aligned}$$

giving the result.

(c) Inserting $f(x)$, and noting that

$$f(x-t) = \begin{cases} 1 & |x-t| \leq a \\ 0 & |x-t| > a \end{cases}$$

so that the limits of the integral become $t = x - a$ and $t = x + a$, gives

$$u(x,y) = \frac{y}{\pi}\int_{x-a}^{x+a} \frac{1}{t^2+y^2}\,\mathrm{d}t = \frac{1}{\pi}\left(\tan^{-1}\left(\frac{x+a}{y}\right) - \tan^{-1}\left(\frac{x-a}{y}\right)\right).$$

6. (a) Write $u(r,\theta) = R(r)T(\theta)$ and insert in Laplace's equation to give

$$\frac{T}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{R}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right) = 0,$$

$$\text{or}\quad \frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = -\frac{1}{T\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right) = \lambda \quad \text{separation cons.}$$

giving the two equations

$$\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right) + \lambda\sin\theta\, T = 0$$

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - \lambda R = 0.$$

Transform the first equation by writing $z = \cos\theta$, $w(z(\theta)) = T(\theta)$. The chain rule then gives

$$\frac{dT}{d\theta} = \frac{dw}{dz}\frac{dz}{d\theta} = -\sin\theta\frac{dw}{dz}$$

$$\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right) = -\frac{dz}{d\theta}\frac{d}{dz}\left(\sin^2\theta\frac{dw}{dz}\right) = \sin\theta\frac{d}{dz}\left((1-z^2)\frac{dw}{dz}\right),$$

which on writing $\lambda = \nu(\nu+1)$ transforms the equation to

$$\frac{d}{dz}\left((1-z^2)\frac{dw}{dz}\right) + \nu(\nu+1)w = 0.$$

The solution must be regular at the poles $\theta = 0, \pi$ or equivalently w must be finite at $z = \pm 1$, providing the b.c.s for a singular Sturm-Liouville problem. Using information given, we must have $\nu = k$ integer, and $w(z) = \mathrm{P}_k(z)$, or $T_k(\theta) = \mathrm{P}_k(\cos\theta)$. The R-equation is now

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - k(k+1)R = 0,$$

which can be solved using the ansatz $R(r) = r^p$ leading to $p(p+1) - k(k+1) = 0$ or $p = k, -(k+1)$, hence $R_k(r) = A_k r^k + B_k/r^{k+1}$ (A_k, B_k constants). This allows the general solution to be written

$$u(r,\theta) = \sum_{k=0}^{\infty} T_k(\theta)R_k(\theta) = \sum_{k=0}^{\infty}\left(A_k r^k + \frac{B_k}{r^{k+1}}\right)\mathrm{P}_k(\cos\theta) \quad \text{as required.}$$

(b) The solution must be finite at $r = 0$ hence $B_k = 0$ for all k. The boundary condition at $r = 1$ gives

$$\sum_{k=0}^{\infty} A_k \mathrm{P}_k(\cos\theta) = T_0\cos\theta\sin^2\theta = T_0(\cos\theta - \cos^3\theta).$$

Using Rodrigues to obtain $\mathrm{P}_1(x) = x$, $\mathrm{P}_3(x) = \frac{1}{2}(5x^3 - 3x)$. Equating sides $A_k = 0 \; \forall k$ except $k = 1, 3$ and

$$A_1 \cos\theta + A_3 \left(\tfrac{1}{2}(5\cos^3\theta - 3\cos\theta)\right) = T_0(\cos\theta - \cos^3\theta),$$

giving $A_3 = -2T_0/5$, $A_1 = 2T_0/5$, and

$$u(r,\theta) = \frac{2T_0}{5}\left(r\mathrm{P}_1(\cos\theta) - r^3\mathrm{P}_3(\cos\theta)\right).$$